Machine Learned Calabi-Yau Metrics and Curvature

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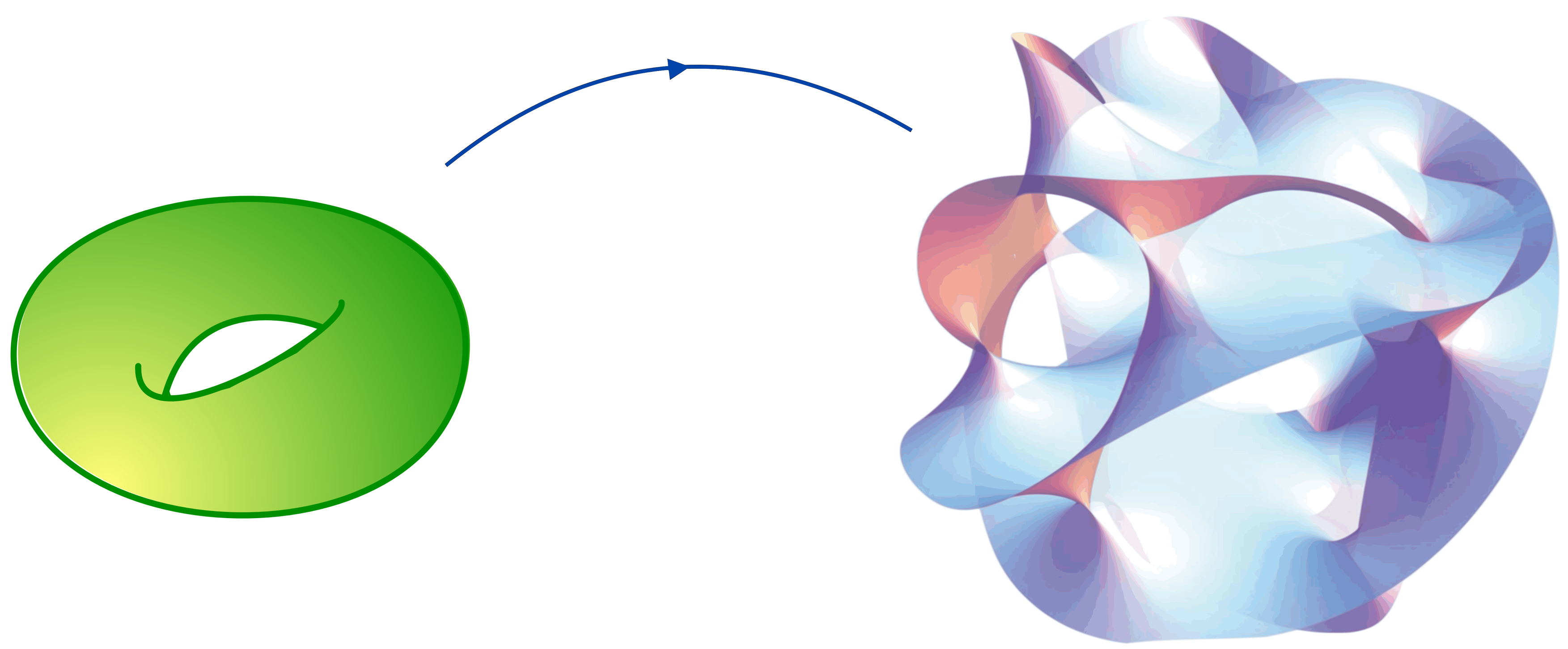
Compactification



$$\{\text{Spacetime}\} \overset{?}{\simeq} \mathbb{R}^{1,3} \times K_D$$

- Geometry/topology of 'compactified' space K_D determines physics of observable space $\mathbb{R}^{1,3}$.
- Many geometric/physical questions intimately tied to the metric tensor g_{ab} on a manifold, analytic form for general K_D unknown.
- Physical considerations suggest K_D is a six-dimensional complex Kähler manifold with a Ricci flat metric, $R_{ab}[g] = 0$ - a **Calabi-Yau threefold** CY_3.
- Informally, to preserve conformal invariance at the quantum level, the metric should be 'scale'-invariant, where μ denotes some energy scale. To leading order:

$$\beta_{ab}[g] \triangleq \frac{\partial g_{ab}}{\partial \log \mu} \sim R_{ab} \implies R_{ab} = 0 \qquad (1)$$

- $\exists$ **many** Calabi-Yau threefolds - each corresponds to a different 'string theory', each yields different low-energy physics in $\mathbb{R}^{1,3}$. Currently have no means to decide which is 'right'.

Approximation

- Variational ansatz for metric: $\tilde{g}_{ab}(\cdot\,;\theta)$.

$$\tilde{g} : (\{p\} \in \text{CY}_n, \text{ geometric/moduli data}, \theta) \approx \{g_{ab}|_p\}_{a,b=1,\ldots D}$$

- θ determined by minimization of some variational functional which enforces Eq. 1.

$$\theta = \underset{\theta' \in \Theta}{\text{argmin}}\, \mathscr{L}(\theta'),\ \theta \in \Theta \subset \mathbb{C}^D, D \gg 1.$$

- Great freedom in the form of $\tilde{g}, \mathscr{L}$. The latter captures known mathematical properties of the (unique) true metric.
- Generally use a 'dd^c-lemma' ansatz. Find $\phi \in C^\infty(\text{CY}_3)$ ($\iota : \text{CY}_3 \hookrightarrow \mathcal{A}$, $\mathcal{A}$ ambient).

$$(\omega + dd^c\phi)^n = e^f \omega^n,\ f \in C^\infty(\text{CY}_3),\ \ \omega = \iota^* \omega_{\text{FS}}\,,$$

where ω_{FS} denotes the ambient Fubini-Study metric on $\mathcal{A}$.
- Metrics in a given cohomology class parameterized by scalar functions on the manifold, in turn parameterized by θ via a neural network, $\varphi_{\text{NN}}(\cdot;\theta)$:

$$\tilde{g} = \iota^* g_{\text{FS}} + \partial\bar{\partial}\varphi_{\text{NN}}$$

Geometry / Topology

- Geometric quantities computed via automatic differentiation w.r.t. CY coords:

$$\varphi \xrightarrow{\partial} (g_{\mu\nu} \sim \partial\partial\varphi) \xrightarrow{\partial} (\Gamma^\kappa_{\mu\nu} \sim g\cdot\partial g) \xrightarrow{\partial} \left(R^\kappa{}_{\lambda\mu\nu} \sim \partial\Gamma + \Gamma\cdot\Gamma\right) \to \cdots$$

- The Chern-Gauss-Bonnet theorem relates local curvature information (gleaned from the curvature two-form $\mathcal{R}$) to global topological data of the manifold X. The Chern classes c_j;

$$\det\left(\mathbb{I} + \frac{i}{2\pi}\epsilon\mathcal{R}\right) = c_0 + c_1\epsilon + c_2\epsilon^2 + \cdots\,,$$

enable computation of topological invariants such as the Euler characteristic,

$$\chi(X)_{\text{E}} = \frac{1}{(2\pi)^n}\int_X c_n(\mathcal{R})\,, \quad \mathcal{R}^\mu{}_\nu = R^\mu{}_{\nu\kappa\lambda}dz^\kappa \wedge d\bar{z}^\lambda.$$

- Sanity check: extract curvature information from approximate metric to reproduce known topological data. Consistency is vital for phenomenology; physically important data, e.g. Yukawa couplings are also global quantities, $\kappa = \int_X a \wedge b \wedge c,\ a,b,c \in H^1(TX)$.
- Differentiable computation of geometric/topological data in `Jax` enables usage in objective function during optimization.

Some algebraic geometry

- Calabi-Yau spaces may be realized as surfaces embedded in complex projective space $\mathbb{P}^n$, e.g. the one-parameter deformation family of quartics:

$$\mathbb{P}^3 \supset X_\lambda := \{p_\lambda(z) = 0\} : \quad p_\lambda(z) := \sum_{i=0}^3 z_i^4 - \frac{\lambda}{3}\left(\sum_{i=0}^3 z_i^2\right)^2. \qquad (2)$$

- Variation of λ parameterizes deformation of the **complex structure** (loosely, the 'shape') of X_λ, which singularizes over the set $\lambda^\# \in \{\frac{3}{4}, 1, \frac{3}{2}, 3\}$. We numerically investigate singular X_λ.
- λ is an example of a complex structure **moduli parameter**. Good control of g_{ab} at arbitrary points in moduli space is important for studying stringy topology-changing processes ('flops').

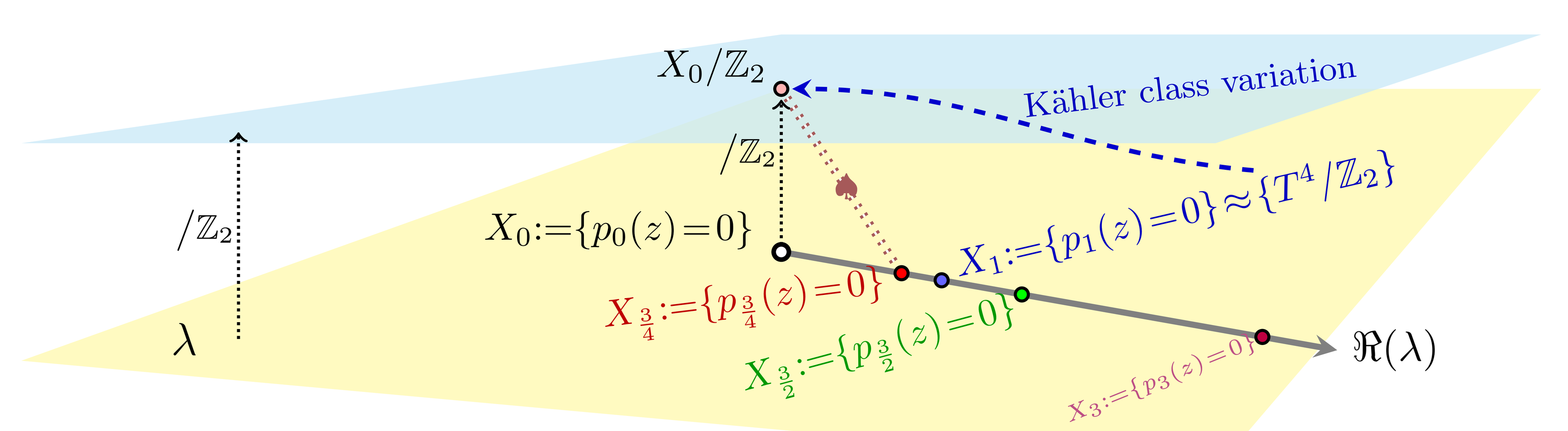


Figure 1. Deformation family of quartics (lower plane), parameterized by $\lambda \in [0, \infty)$, and its $\mathbb{Z}_2$ quotient (upper plane). Each X_λ corresponds to a distinct Calabi-Yau manifold. Points where X_λ develops singularities are marked.

- Numerically investigating topological quantities on singular X_λ allow us to postulate + prove:

Proposition: Let $X_s \subseteq X_\lambda \subseteq \mathbb{P}^3$, the smooth locus of singular X_λ, with associated curvature two-form $\mathcal{R}$. Denote the Fulton class of X_λ by c_F. If $|\text{Sing}X_\lambda| < \infty$, then:

$$\frac{1}{(2\pi)^2}\int_{X_s} c_2(\mathcal{R}) + |\text{Sing}X_\lambda| = \deg c_F(X_\lambda)\,. \qquad (3)$$

Numerics

- Studying singular X_λ, we numerically observe + conjecture:

λ	$\lvert\text{Sing}(X_\lambda)\rvert$	$\deg c_2(\mathcal{R})$	$\deg c_1(\mathcal{R})^2$
0	0	24	0
3/4	8	7.99 ± 0.03	-16.0 ± 0.2
1	16	-7.99 ± 0.08	-31.9 ± 0.3
3/2	12	0.0 ± 0.1	-23.9 ± 0.3
3	4	16.00 ± 0.09	-8.0 ± 0.1

Table 1. Values of Monte Carlo integrals of the possible top characteristic forms on X_λ, $\pm$ 2 std. dev.

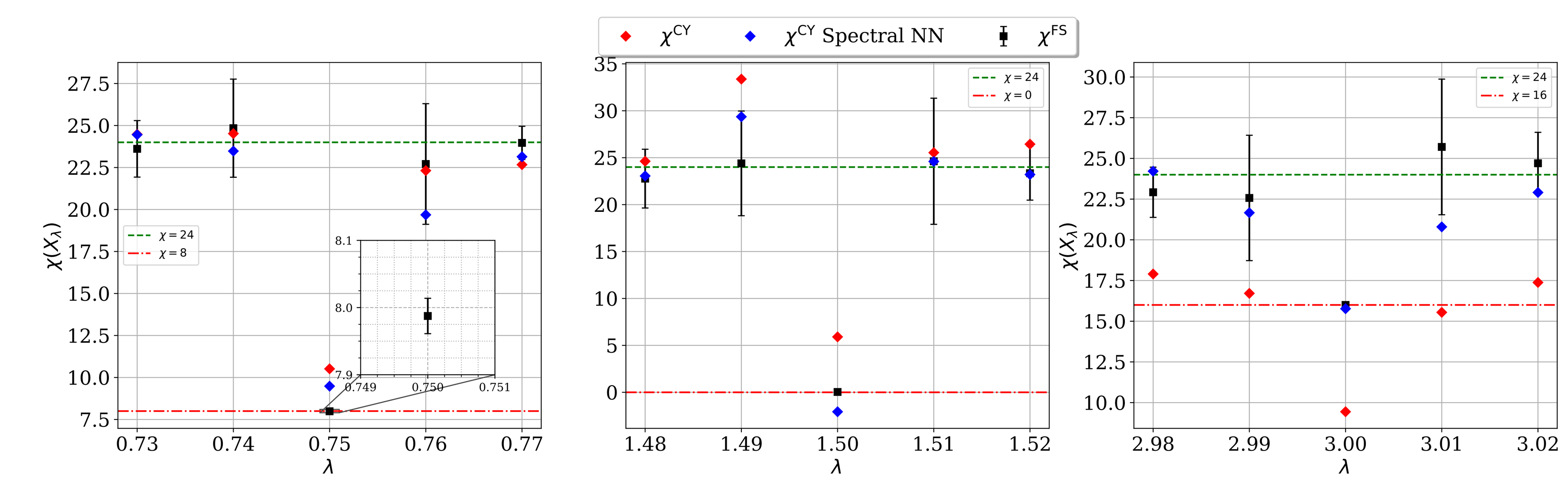


Figure 2. Numerical values of $\int_{X_\lambda} c_2$ across different regions of moduli space. Red (blue) dots arise from learned metric using fully-connected (spectral) networks. Black dots arise from pullback of the Fubini-Study metric on $\mathbb{P}^3$.

Conjecture: Let $X \subseteq \mathbb{P}^3$ be a possibly singular K3 surface, whose smooth locus has curvature form $\mathcal{R}$. If the singularities of X are isolated, then:

$$\int_{X_s} c_2(\mathcal{R}) - (c_1 \wedge c_1)(\mathcal{R}) = 24 = \chi(\text{K3})\,. \qquad (4)$$

- For crepant (c_1-preserving) resolution of singular M with discrete group action G, fixed point set F, desingularizing surgical replacement N, the Euler characteristic is:

$$\chi(M/G) = \frac{1}{|G|}(\chi(M) - \chi(F)) + \chi(N)\,, \qquad (5)$$

here $|G = \mathbb{Z}_2| = 2$ and N consists of $|\text{Sing}(X_\lambda)|$ isolated exceptional $\mathbb{P}^1$-like divisors.
- The $\deg c_1(\mathcal{R})^2$ column in Table 1 yields the sum of the isolated contributions with $\deg c_1^2 = -\chi(N) = -2$, and the $\deg c_2(\mathcal{R})$ column gives the leading term in Eq. 5. Motivating,

Conjecture: Each $X_{\lambda^\#}$ with $\lambda^\# < \infty$ may be identified with a global finite quotient Y/G.
- Strangely, the learned metric appears to 'know' enough algebraic geometry to perform a crepant resolution for singular X_λ.

Outlook

- View the metric g_{ab} as a function's-worth of couplings in the stringy (bosonic) nonlinear-σ model (NLσM) action:

$$S \sim \int d\text{Vol}\, h^{\alpha\beta}\partial_\alpha X^a \partial_\beta X^b g_{ab}\,.$$

Our procedure reproduces the outcome of the approximate renormalization group (RG) flow of g_{ab} (Eq. 1) - can we generalize this to the RG flow of general NLσMs?
- RG flow $\to$ exotic heat flow $\to$ gradient flow $\overset{?}{\to}$ optimal transport? $\to$ numerics.
- Simulations of RG flow may lead to non-perturbative approximation methods for investigating conformal field theories in the strong coupling regime.